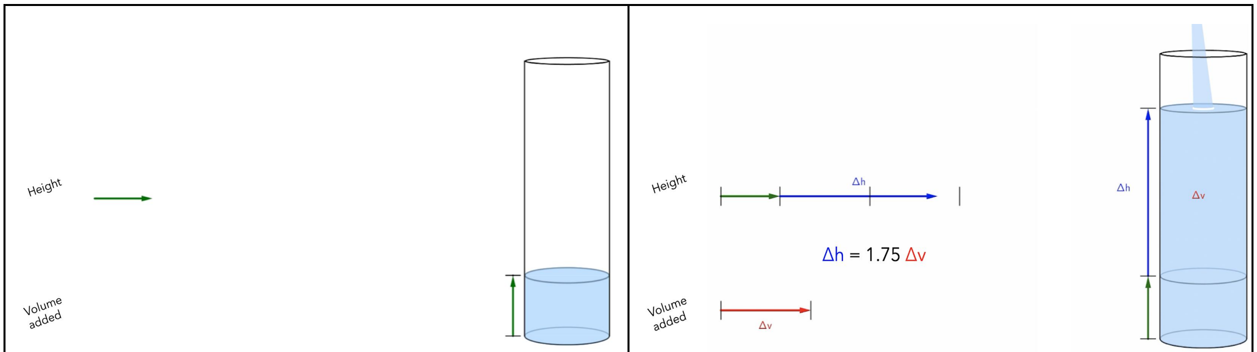
# <u>Concepts and Terminology Used in our Videos</u>

Our development of the core ideas of calculus is based on the idea of coordinating amounts of change of co-varying quantities. We define the derivative in terms of a limit of average rates of change of quantities, and this average rate is grounded in the idea of a constant rate, which is in turn based on the idea of a proportional relationship between the amounts of change of two quantities. These same ideas of constant rates of change are used to ground the idea of Riemann sums and definite integrals as accumulation, and provide a natural connection between rates of change and amounts of accumulation—i.e., the fundamental theorem of calculus.

This document defines and describes these grounding concepts and the related terminology, and identifies the notation we use in the development of these concepts in our videos.




A. Cylinder with initial amount of water

B. Cylinder filling with water

*Figure 1. Images from the first video on Constant Rate of Change depicting quantities for change in height and change in volume of the water in the cylinder.*

***Quantity:*** Measurable attribute of an object or situation. A clear description of a quantity includes the following:

- A brief reference to the object or situation,
- The attribute being measured,
- Where the quantity is measured from,
- The direction of measurement, and
- The units used in the measurement.

For example, attributes of a cylinder being filled with water that are quantities include the initial height of the water (Figure 1A) and the change in height of the water (Figure 1B) both measured in inches, and the initial volume of water and the added volume of the water (Figure 1B) both measured in cubic inches. How much you may like the taste of the water is not a quantity. We often use vectors to depict quantities (Figure 1) because the magnitude of the vector can be measured and a vector's direction can indicate the direction of measurement.

***Fixed vs Varying Quantities:*** If the value of a quantity does not change, then the quantity is called ***fixed*** and its value is constant. If the value of a quantity changes, then the quantity is called ***varying*** and assumes more than one value.

For example, in Figure 1A, the cylinder starts with some water already in it before additional water is added. So the initial volume in cubic inches and the initial height of the water in the cylinder measured in inches are fixed. Distinctions can also be made between attributes of the water in the cylinder and attributes of the cylinder. The height, radius, diameter, and weight of the cylinder are all fixed quantities.

***Variable:*** A designating letter or symbol to represent the values that a specific varying quantity can assume.

Students' experiences with variables are often dominated by repeatedly solving for $x$. Thus, many students routinely think of variables, such as $x$, as something to solve for. This definition of variable pushes students to think beyond merely solving for $x$ and better supports modeling tasks found throughout calculus and STEM fields.

***Amount of Change:*** A change in a quantity's value as the quantity varies over an interval of values. The change in a quantity's value is a new quantity itself and is often denoted using $\Delta$ notation.

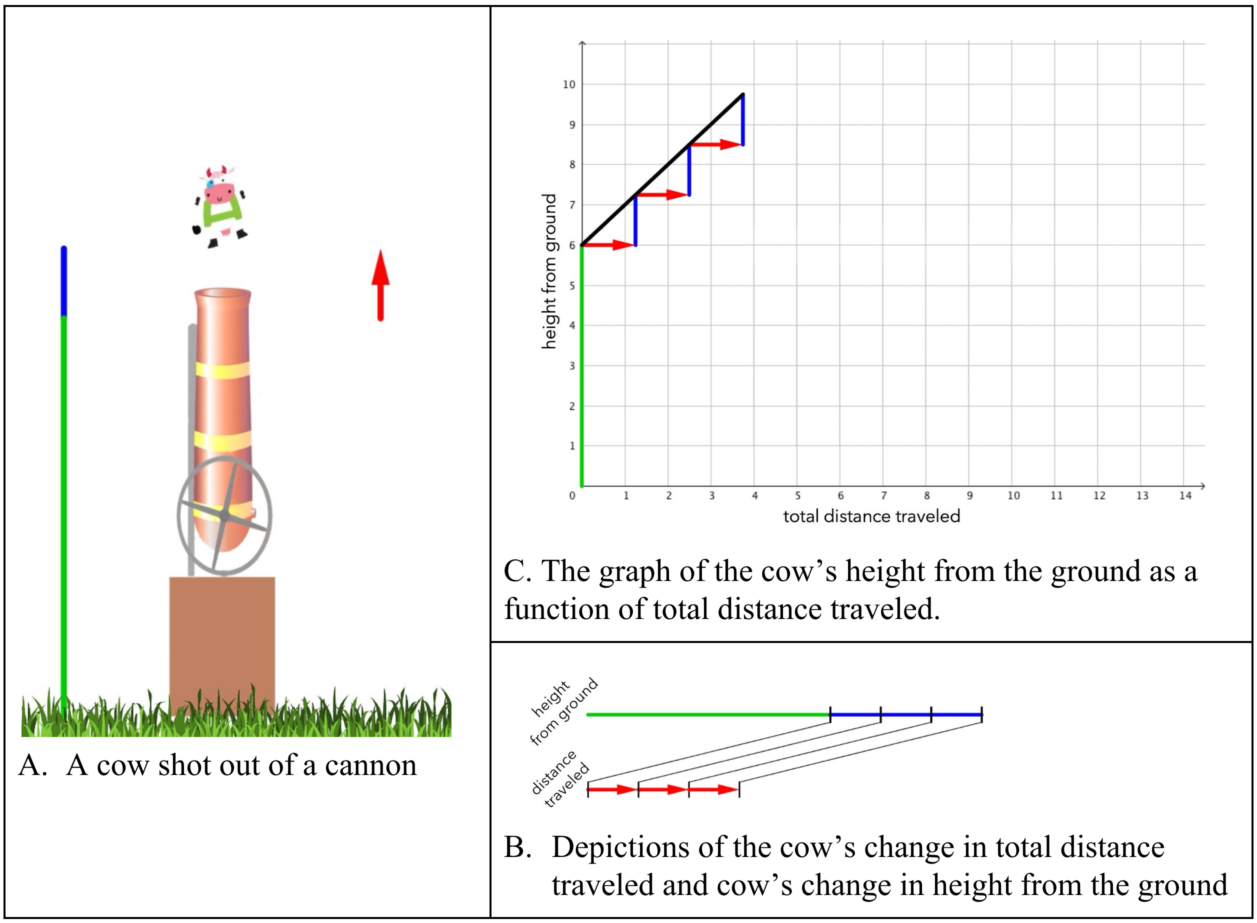
For example, in Figure 1B, the amount of change in height and the amount of change in volume are denoted using $\Delta h$ and $\Delta v$, respectively. In addition, these amounts of change are depicted by the magnitudes of corresponding vectors. Had water been taken out of the cylinder, then the vectors would have been pointing in the opposite direction.

***Proportional:*** When the ratio of two varying quantities is constant, we say that the ***quantities are proportional***.

For example, in the first video for Constant Rate of change, as the cylinder is filling with water, the relationship, $\Delta h = 1.75\Delta v$, is invariant (Figure 1B), meaning that the amount of change in height is proportional to the amount of change in volume.

***Constant Rate of Change:*** Two quantities ***change at a constant rate*** with respect to each other if changes in one quantity are proportional to corresponding changes in the other. This means that if a function $f$ from $x = x_1$ to $x = x_2$ changes at a constant rate $m$, then this relationship can be expressed as $\Delta f = m \Delta x$, which emphasizes that $\Delta f$ is $m$ times as large as $\Delta x$ and is equivalent to $m = \Delta f / \Delta x$.

> ***Consequence of the Constant Rate of Change Definition:*** For every fixed amount of change in the independent quantity, the amount of change of the dependent quantity remains constant.




A. A cow shot out of a cannon

C. The graph of the cow's height from the ground as a function of total distance traveled.

B. Depictions of the cow's change in total distance traveled and cow's change in height from the ground

*Figure 2. Images from the first video on Graphing Constant Rate of Change depicting quantities for change in height from the ground and change in distance traveled.*

In the scenario of a cow shot out of a cannon (Figure 2A), during the cow's trip up, the cow's change in height from the ground is always 1.0 times as large as the cow's change in distance traveled. So the graph of the cow's height off the ground as a function of distance traveled during the cow's trip up is the black line seen in Figure 2C. In addition, for every fixed amount of change in distance traveled (the length of the red vectors in Figures 2B and 2C), the amount of change in height (the length of the blue line segments in Figures 2B and 2C) remains constant.

***Increasing & Decreasing Rate of Change:*** Two quantities have an ***increasing rate of change*** with respect to each other if for every fixed amount of change in the independent quantity, the amount of change of the dependent quantity is increasing. Two quantities have a ***decreasing rate of change*** with respect to each other if for every fixed amount of change in the independent quantity, the amount of change of the dependent quantity is decreasing.

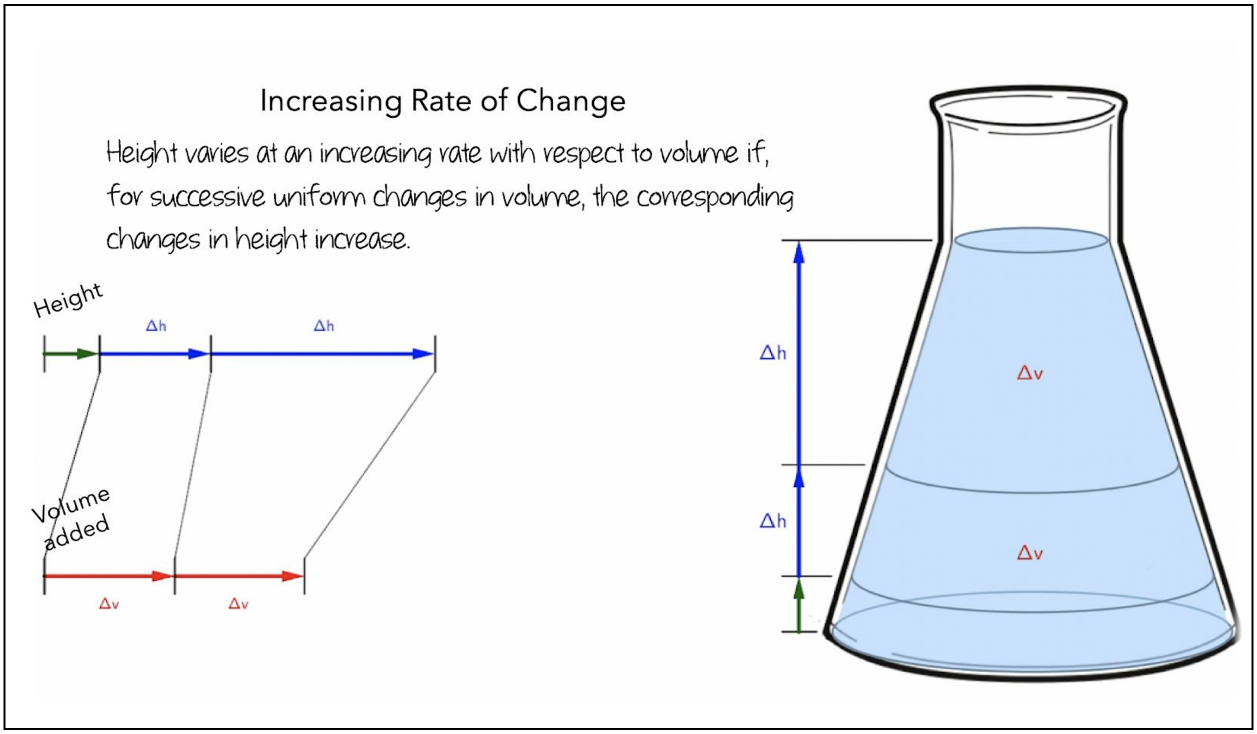
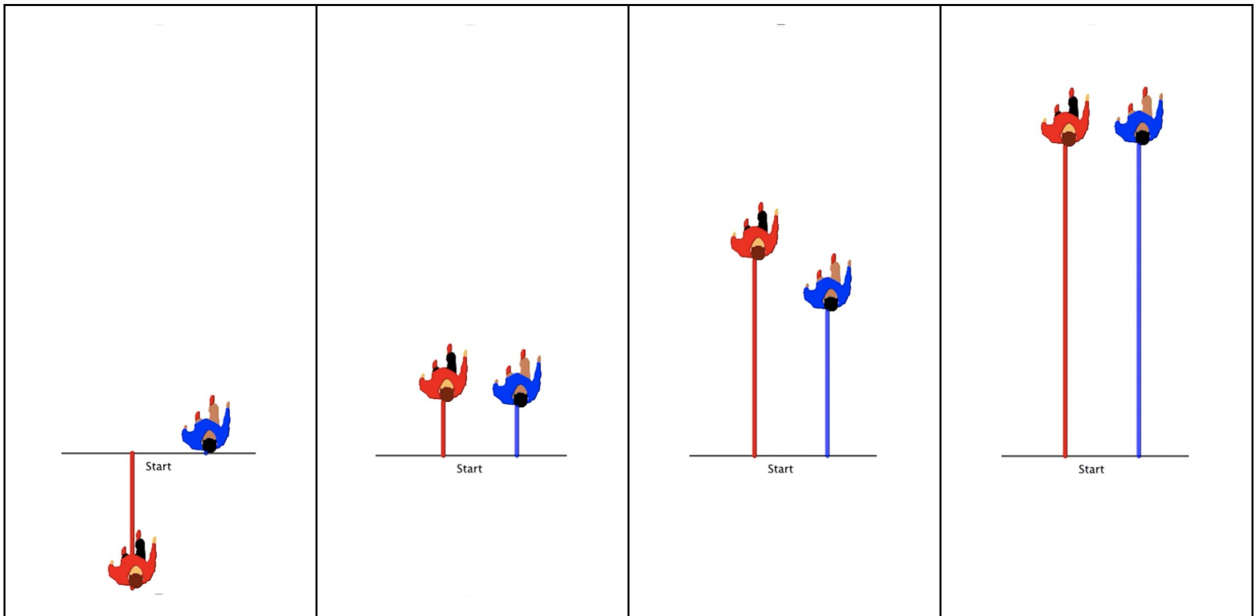
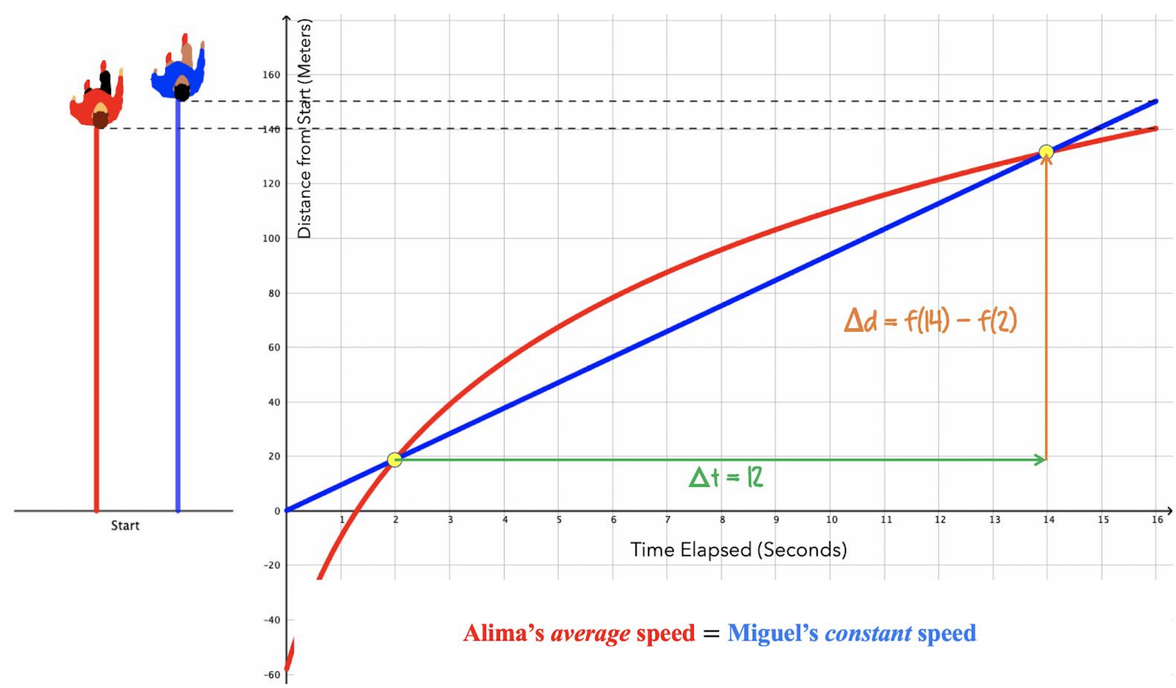



*Figure 3. Screenshot from the first video on Increasing Rate of Change depicting quantities for change in height and change in volume of the water in the flask.*

***Average Rate of Change:*** The ***average rate of change*** of a function $f$ from $x = x_1$ to $x = x_2$ is the constant rate of change of a linear function $g$ that has the same change in output as the function $f$ over the interval $[x_1, x_2]$.




A. Tracking Alima’s and Miguel’s distance from the starting line as Alima speeds up and slows down compared to Miguel running at a constant pace.




B. A graph highlighting Alima’s and Miguel’s change in distance over the same change in time from 2 seconds to 14 seconds.

*Figure 4. Images from the second video on Average Rate of Change depicting amounts of change for relevant quantities for two different runners, one running at a constant speed and one speeding up and slowing down, as both runners pass the starting line.*

In the scenario a runner running laps around a track (Figure 4), a runner's (Alima in Figure 4) average speed over an interval of time (i.e., 2 seconds to 14 seconds) can be viewed as the constant speed (the slope of the blue line in Figure 4B) needed by another runner (Miguel in Figure 4) to cover the same distance (depicted by the length of the orange vector in Figure 4B) in the same amount of time (depicted by the length of the green vector in Figure 4B).

***Instantaneous Rate of Change:*** An average rate of change over an interval so small that changes in the quantities' measures are essentially proportional.

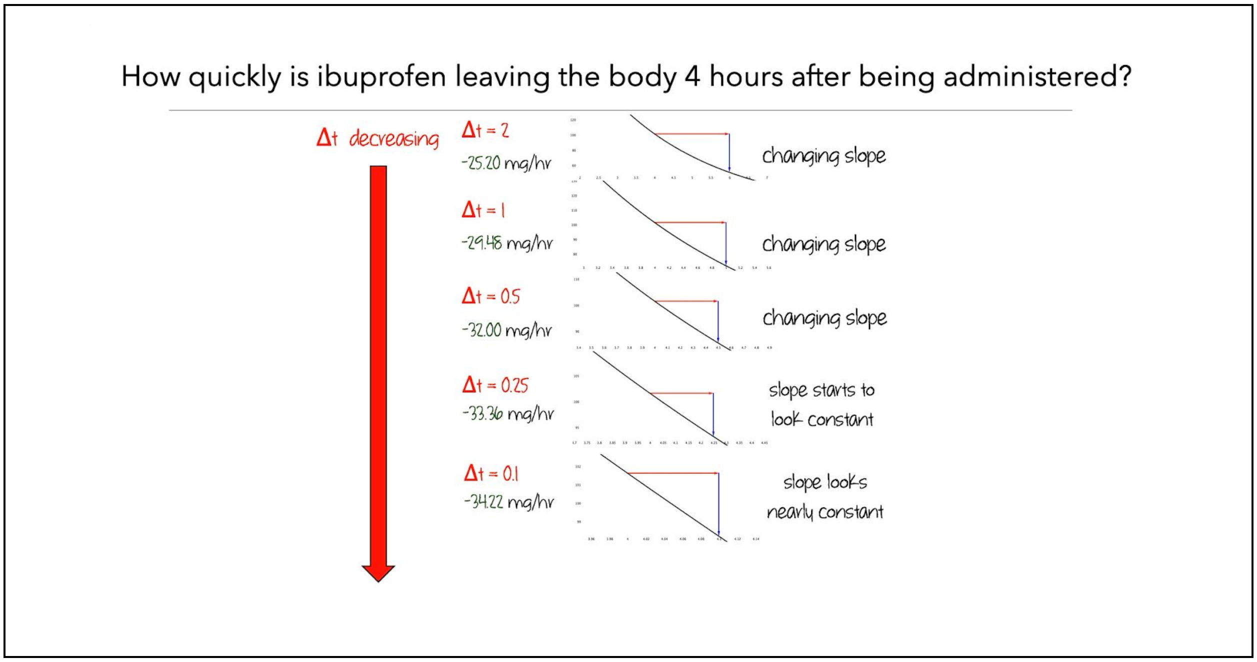



*Figure 5. Screenshot from the Limit Definition of the Derivative video progressively zooming through five graphs depicting how the amount of change in ibuprofen in a body becomes essentially proportional to the amount of change in time since administered.*

From top to bottom, the five graphs in Figure 5, are graphs of the amount of ibuprofen in a body as a function of time after being administered. These graphs all focus on 4 hours since administered but indicate decreasing values of the amount of change in time. This supports students in observing that the slope of the function "looks nearly constant" and constant slope has previously been associated with changes in quantities measures being proportional (Figure 2C).

***Approximation Language:*** When unpacking concepts defined in terms of limit, such as derivative and definite integral, we typically take an error analysis approach and highlight how approximations can be made as accurate as desired by decreasing the error. This approach has proven consistent with formal limit definitions while simultaneously leveraging and building upon students' intuitions. This approach also supports a more coherent approach to introduce concepts defined in terms of limit (derivative and definite integral from first-semester calculus). The following table details how approximations and error analysis informed our videos.

| Guiding Questions | Terminology Introduced | Description | Derivative Example | Integral Example |
|---|---|---|---|---|
| What is being approximated? | Unknown Value | What you are approximating. If the value was known, then approximations would not be needed. | An instantaneous rate of change.<br>Notation: $f'(a)$<br>Graphically: The slope of a tangent line at $x = a$ . | A total accumulation.<br>Notation: $\int_a^b r(x)dx$<br>Graphically: The area "under" the rate curve for a positive valued function. |
| What are the approximations? | Overestimates and Underestimates | Estimates to the unknown value. An overestimate would have value greater than the unknown value. An underestimate would have value less than the unknown value. | An average rate of change.<br>In general, $f'(a) \approx \frac{\Delta f}{\Delta x} = \frac{f(a+\Delta x)-f(a)}{\Delta x}$.<br>When rate is decreasing (for example), Overestimate = $\frac{f(a+\Delta x)-f(a)}{\Delta x}$ for $\Delta x < 0$, and Underestimate = $\frac{f(a+\Delta x)-f(a)}{\Delta x}$ for $\Delta x > 0$.<br>Graphically: The slopes of secant lines. | A Riemann sum.<br>In general, $\int_a^b r(x)dx \approx \sum_{k=1}^{n} r(x_k)\Delta x$ .<br>When rate is decreasing (for example),<br>Overestimate = $\sum_{k=0}^{n-1} r(x_k)\Delta x$ , and<br>Underestimate = $\sum_{k=1}^{n} r(x_k)\Delta x$ .<br>Graphically: The sum of areas of rectangles. |
| How good is my approximation? | Error | The difference between the unknown value and an approximation. The value of error will also be unknown. | Difference between the desired instantaneous rate and an average rate of change.<br>Graphically: The difference in the slope of the tangent in and a slope of a secant line. | Difference between the total accumulation and the value of a Riemann sum.<br>Graphically: The sum of the difference between the area "under" a rate curve and the area of rectangles. Often appears similar to the sum of areas of "triangles". |
| Can I make my approximations more accurate? | Decrease error | The closer error is to zero, the better the approximations. | Use smaller $\Delta x$ . | Use smaller $\Delta x$ . |
| How do I guarantee a desired accuracy? | Error Bound | Simply a bound on the error, but operationalized as the difference between an overestimate and an underestimate. Make the difference between the overestimate and the underestimate less than the desired error bound. | A difference between average rates of change.<br>Graphically: A difference between slopes of secant lines. | A difference between Riemann sums.<br>Graphically: A difference between rectangles "above" the rate curve and "below" the rate curve. |

***Infinitesimal Language:*** Within the videos, the "desired accuracy" is often based on visual cues, and language such as "essentially proportional" and "indistinguishable" appear alongside uses of "small" and "sufficiently small." This language could lead students to believing that error must become zero for the value of a limit to exist. To counteract this, we often indicate multiple examples of a desired accuracy and employ zooming to indicate how error still exists.

***Derivative of a function f at a point a:*** The derivative of a function $f$ at a point $x = a$ is defined as a limit of average rates, $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$ .

***Riemann Sum:*** A Riemann sum is an approximation to the total amount of accumulation of a dependent quantity over an interval of the independent quantity's variation. The value of a Riemann sum is obtained by assuming that rate is constant over successive uniform intervals of the independent quantity's variation. The total amount of accumulation is approximated by the sum of accumulations over these intervals. Smaller uniform intervals over which rate is assumed constant can yield better approximations to the total accumulation.

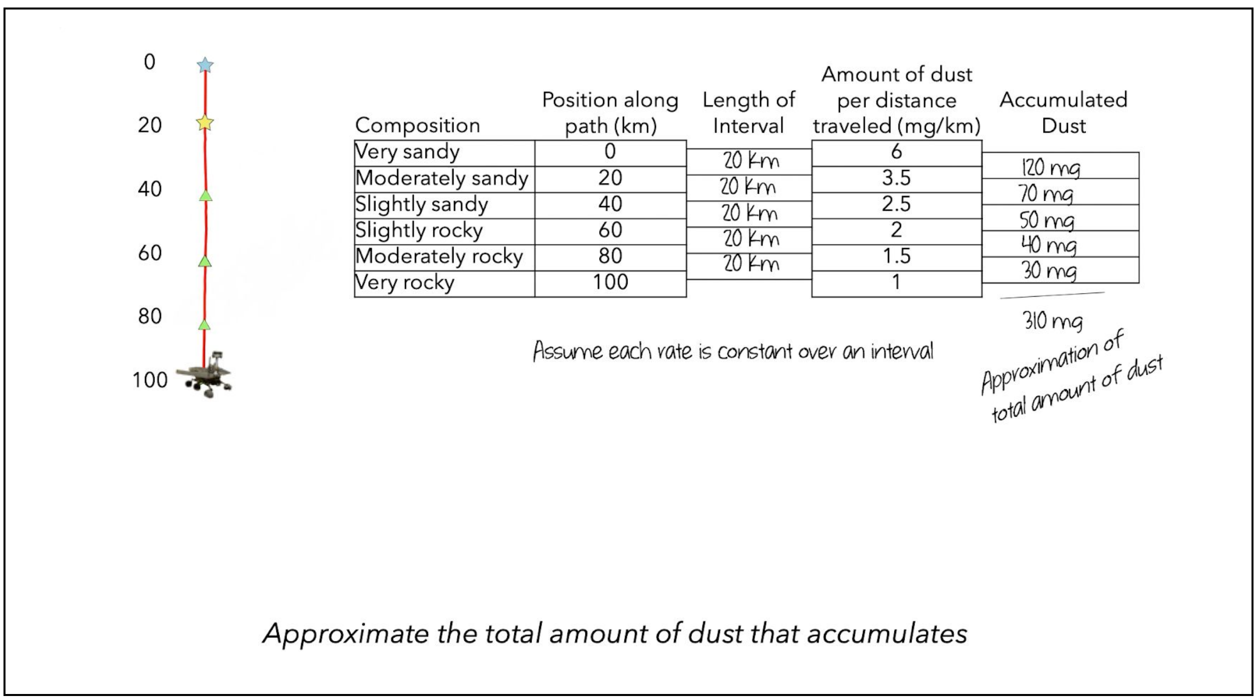



*Figure 6. Screenshot from the first video on Riemann Sums where the viewer is asked to approximate the total amount of dust that accumulates on the solar panels of a Mars rover given different rates of dust accumulation per distance traveled based on the composition of the Martian surface under the rover.*

To approximate the dust accumulation on the Mars rover solar panels (Figure 6), the rate of dust accumulation per distance traveled is assumed constant over each 20km interval of the rover's path. In particular, in Figure 6, the largest rate of change over each interval is used to approximate the dust accumulated over that interval. Then each approximation over each interval is added to approximate the total amount of dust accumulated over the entire path.

To support the development of this reasoning, initially presenting Riemann sums depicted as a sum of areas of rectangle was avoided to encourage students to focus on the relevant quantities. When depicted using rectangles, the meaning of the rectangle's area as an approximation to an accumulating quantity where rate is assumed constant over an interval of the independent quantity's variation can be obfuscated by the more salient attributes of the geometric image (rectangles with shaded in areas). Later Riemann sums are depicted using the sum of the areas of rectangles (i.e., see Figure 8).

***Introduction of the Index:*** Assuming a constant rate = $R(x)$ over successive uniform intervals of the independent quantity's variation, a Reimann sum is given by $\sum R(x) \cdot \Delta x$. This notation is introduced early but it does not distinguish between left and right Riemann sums nor does it indicate the number of successive uniform intervals. The index $k$ and an upper bound on $k$, $n$, is introduced to make this distinction, $\sum_{k=1}^{n} R(x_k)\,\Delta x$. See Figure 7 for an example.

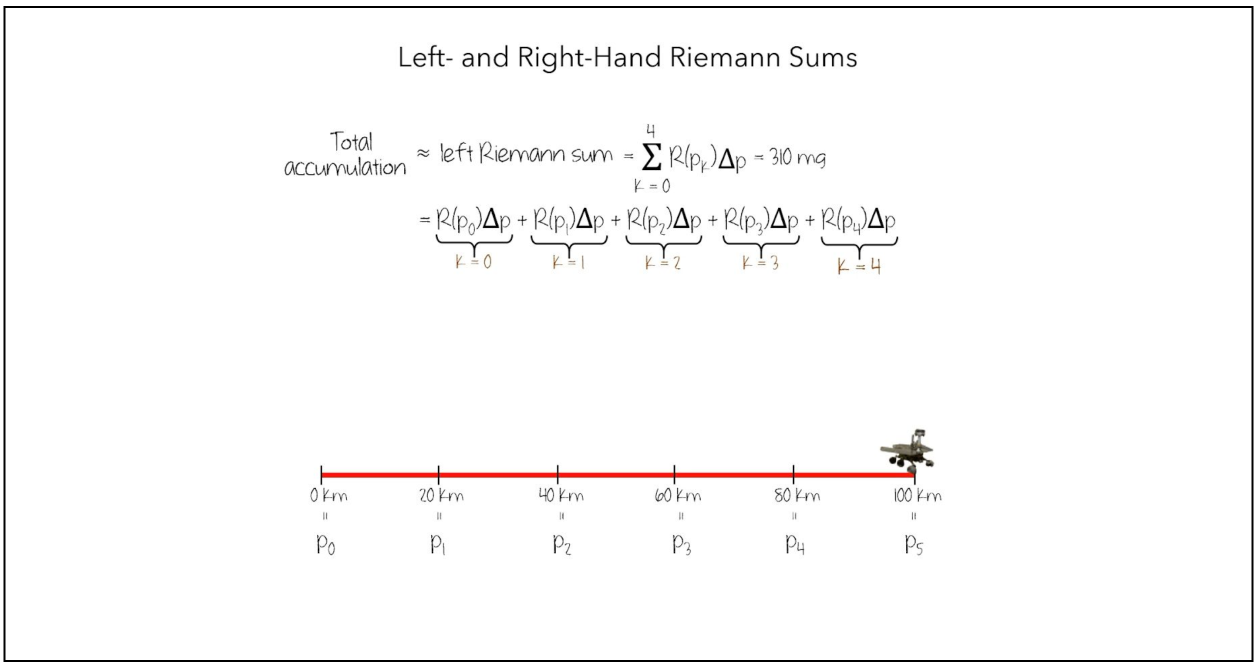
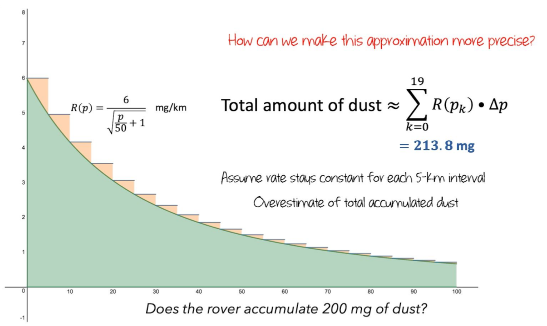


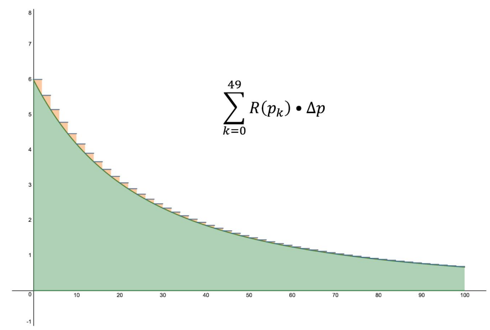

*Figure 7. Screenshot from the Riemann Sum Notation that introduces the index using summation notation.*

***Definite Integral:*** The exact total accumulation.

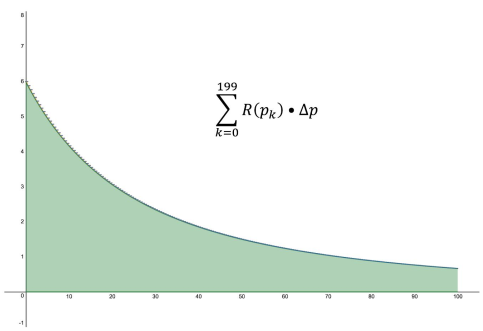
***Consequence of the Riemann sum and Definite Integral Definition:*** The definite integral is the change in values of the antiderivatives over the interval of the independent quantity's variation.



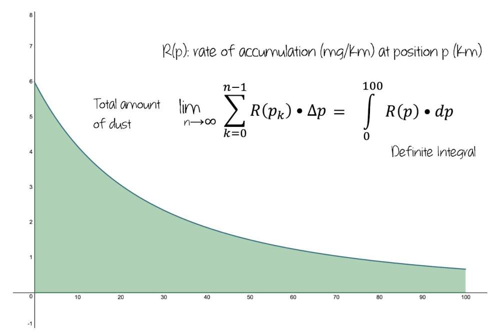

A. An approximation with error. Includes a question that leads to making the approximation more precise.




B. Making more precise by increasing the number of divisions of the interval over which the independent quantity varies. Error can be seen.



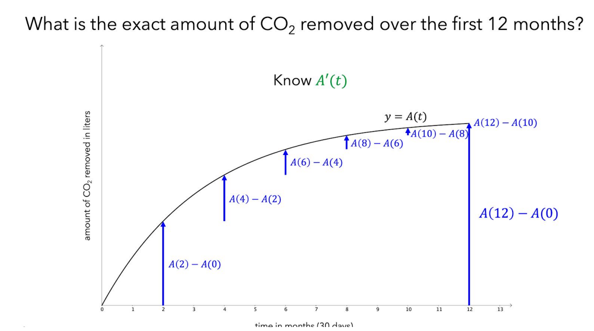

C. Another increase to the number of divisions of the interval the interval over which the independent quantity varies. Error is less but still can be seen.



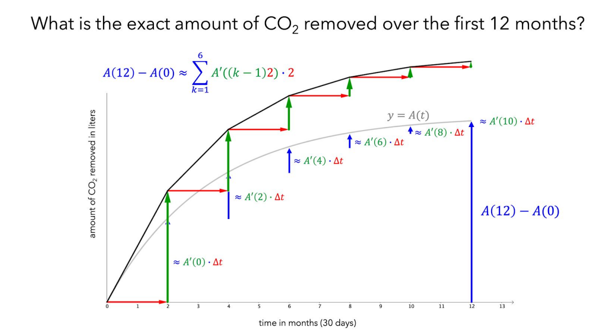

D. A definite integral defined as the limit of the Riemann sums which would give the exact total accumulation. Error is no longer visible.

*Figure 8. Screenshots from the first video from Definite Integrals depicting improving approximation using Riemann Sums where rate is assumed constant over smaller and smaller intervals until error is essentially zero.*

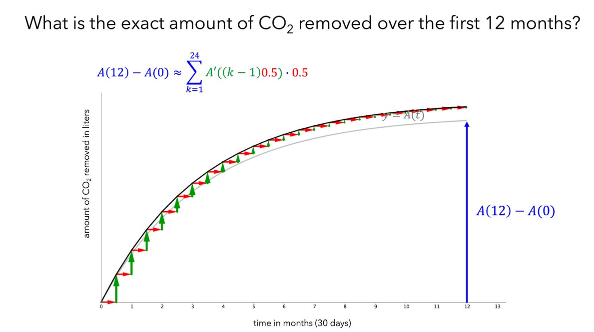
The exact total accumulation of the amount of dust on the Martian rover (assuming a related equation, Figure 8A), can now be depicted as the exact area under the graph of the corresponding rate function (green area in Figure 8). Assumed constant rates over small intervals of the independent quantity are depicted graphically as pieces of piecewise constant functions (Figure 8A, 8B, 8C). The difference between the exact total accumulation and any one approximation is depicted as the sum of the areas of “triangle like” light orange regions on the graph (see best in Figure 8A and 8B but also barely visible in Figure 8C).



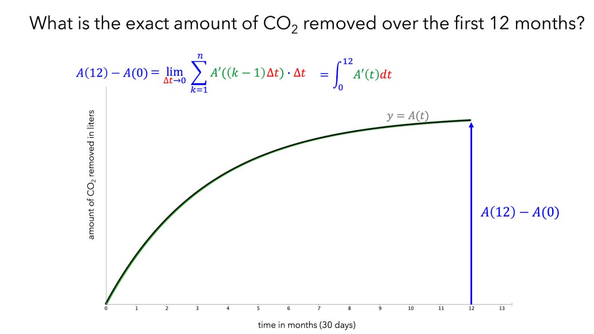

A. Actual amounts of change of the dependent quantity depicted on the “amount” function.




B. Graphing an approximation to the amount “function” using known rates.




C. Improving the approximation to the “amount” function with more divisions of the interval over which the independent quantity varies.




D. Exact amount of change of the dependent quantity over variation of the independent quantity as a definite integral.

*Figure 9. Screenshots from the First Fundamental Theorem of Calculus video (Integrals are Antiderivatives) where Riemann Sums are depicted on an “amount” function.*

The First Fundamental Theorem of Calculus (Integrals are Antiderivatives) is an immediate consequence of our definition of the definite integral. Instead of depicting Reimann sums on a “rate of change” graph, in our video on the First Fundamental Theorem of Calculus, Riemann sums are depicted on an “amount” graph (Figure 9). We present a scenario where an air scrubber removes $CO_2$ from an enclosed environment. Again, assuming that rate is constant over small intervals of change of the independent quantity, the overall amount of change of the dependent quantity is approximated by a Riemann sum (Figure 9B and 9C). The assumed constant rates, instead of appearing as a piecewise constant function (Figure 8A, 8B, and 8C), now is depicted more generally as the slopes of a piecewise linear function (Figure 9B and 9C). Thus the limit of the Riemann sum, or definite integral, would yield the exact amount of $CO_2$ removed over the entire elapsed time.